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On selection of repeated unit cell model and application of unified periodic boundary conditions in micro-mechanical analysis of composites

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Abstract

Most micro-mechanical analyses for composites are based on repeated unit cell models (RUCs) by assuming a periodical distribution of the reinforcing phase. In this paper, the uniqueness of solution by applying unified displacement-difference periodic boundary conditions on the RUCs has been proved. Further it is deduced that (1) selection of the RUCs for a fixed periodic array may not be unique, however, the solution is independent on the choice of the different RUCs; (2) boundary traction continuity conditions can be guaranteed by the application of the proposed unified displacement-difference periodic boundary conditions. Illustrative examples are presented and advantages of applying this type of unified periodic boundary conditions are discussed.

Keywords: Micro-mechanics of composites; Repeated unit cell models; Unified periodic boundary conditions; Finite element method

1. Introduction

Composite materials are widely used in advanced structures in astronautics, automobile, marine, petrochemical and many other industries due to their superior properties over conventional engineering materials. In the past several decades many researchers have devoted considerable effort to evaluate macro-mechanical properties of composites by using micro-mechanics modeling method. Micro-mechanical method provides

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overall behavior of the composites from known properties of the reinforcing phase (particles, fiber or fiber yarns) and the matrix phase (polymers, metals) through an analysis of a representative volume element (RVE) (Wolodko et al., 2000) or a unit cell model (Aboudi, 1991; Nemat-Nasser and Hori, 1993). For many composites, such as textile composites, the macrostructure can be seen as a periodic array of a repeated unit cell (RUC). While for some other composites, such as unidirectional lamina, whisker or particulate reinforced composites, a RUC can still be constructed after assuming a uniform distribution and the same geometry for the reinforcing phase. Therefore, in most micro-mechanical analyses the RUC is chosen as the RVE for the composites.

A mathematical presentation of periodic composites, called 'asymptotic homogenization theory', can be found, e.g., in Suquet (1987), Moorthy and Ghosh (1998), Raghavan et al. (2001), among others.

Aboudi (1991) has developed a unified micro-mechanical theory based on the study of interacting periodic cells, and it was used to predict the overall behavior of composite materials both for the elastic and inelastic constituents. In his work and many other references, homogeneous displacement boundary conditions equivalent to the "plane-remains-plane" conditions were applied to the RVE or unit cell models. In fact, the "plane-remains-plane" is only valid for the symmetric RUC subjected to normal tractions. Many researchers, e.g., Needleman and Tvergaard (1993), Sun and Vaidya (1996) and Suquet (1987), have indicated that the 'plane-remains-plane' boundary conditions are over-constrained boundary conditions.

Hori and Nemat-Nasser (1999) presented a universal inequality, which indicates that the predicted effective elastic modulus can vary depending on the applied conditions on the boundary ∂V of a unit cell. And the homogeneous displacement and homogeneous traction boundary conditions will give the upper and lower bounds of the effective modulus. Hollister and Kikuchi (1992) have concluded that the homogenization theory, which uses the periodic boundary conditions, yields more accurate results. Several variational formulations of the homogenization problems for periodic media, with or without orthogonal axes of symmetry and characterized by any geometry of the inclusions, are presented by Luciano and Sacco (1998).

Finite element method (FEM) has been extensively used in the literature to analyze a RUC, to determine the mechanical properties and damage mechanisms of composites. The applications covered from unidirectional laminate (Allen and Boyd, 1993; Bonora et al., 1994; Li, 1999), cross-ply laminates (Bigelow, 1993; Xia et al., 2000; Zhang et al., 2005), to woven and braided textile composites (Dasgupta et al., 1996; Tan et al., 1997; Bystrom et al., 2000). High computer performance in combination with easy-to-use commercial model-creation software (Pro/Engineer, AutoCAD, etc.) and FEM software has contributed to this development. Thus it has become relatively easy to apply FEM to solid RUCs with all levels of complexity. Recently, the parallel finite element approach has been applied to the periodic boundary value problems (Kristensson et al., 2003).

Another important issue in the FEM analysis of RUC is appropriate application of the periodic boundary conditions. As mentioned above, in earlier applications (even some recent ones), "plain-remains-plane" conditions were incorrectly used. There are several publications in which the general periodic boundary conditions have been discussed and were applied to different types of composites, e.g., Li (2000), for unidirectional laminate, Xia et al. (2003), for angle-ply laminates, Tang and Whitcomb (2003), for textile composites. In the latter, the boundary conditions for a sub-unit with certain symmetry in the RUC have also been discussed. In these applications, the applied periodic boundary conditions in the FEM can be expressed as a type of constrained equations of displacement differences on the opposite boundaries of the RUCs.

In this paper, uniqueness of the solution of a periodic boundary value problem with application of the displacement-difference boundary conditions in displacement-based FEM is presented. From the uniqueness of the solution, it can be further deduced that (1) selection of the RUCs for a fixed periodic array may not be unique, however, the solution is independent on the choice of the RUCs; (2) boundary traction continuity conditions can be guaranteed by the application of the proposed displacement-difference periodic boundary conditions. Illustrative examples are presented and advantages of applying this type of unified periodic boundary conditions are discussed.

2. Unified periodic boundary conditions for RUCs

Consider a periodic structure consisting of periodic array of repeated unit cells (RUCs) as shown in the Fig. 1a. The dark area is the reinforcing phase. It is chosen in a trapezoidal shape to represent a more general RUC without any symmetry. The displacement field for the periodic structure can be expressed as (Suquet, 1987)

$$u_i(x_1, x_2, x_3) = \bar{\varepsilon}_{ik} x_k + u_i^*(x_1, x_2, x_3). \tag{1}$$

In the above, $\bar{\varepsilon}_{ik}$ is the global (average) strain tensor of the periodic structure and the first term on the right side represents a linear distributed displacement field. The second term on the right side, $u_i^*(x_1, x_2, x_3)$, is a periodic function from one RUC to another (Fig. 1b). It represents a modification to the linear displacement field due to the heterogeneous structure of the composites.

Since the periodic array of the RUCs represents a continuous physical body, two continuities must be satisfied at the boundaries of the neighboring RUCs. One is that the displacements must be continuous, i.e., the neighboring RUCs cannot be separated or encroach into each other at the boundaries after the deformation. The second condition implies that the traction distributions at the opposite parallel boundaries of a RUC must be the same. In this manner, the individual RUCs can thus be assembled as a physically continuous body.

Obviously, the assumption of displacement field in the form of Eq. (1) meets the first of the above requirements. Unfortunately, it cannot be directly applied to the boundaries since the periodic part, $u_i^*(x_1, x_2, x_3)$, is generally unknown. For any RUC, its boundary surfaces must always appear in parallel pairs, the displacements on a pair of parallel opposite boundary surfaces can be written as

$$u_i^{j+} = \bar{\varepsilon}_{ik} x_k^{j+} + u_i^*, \tag{2}$$

$$u_i^{j-} = \bar{\varepsilon}_{ik} x_k^{j-} + u_i^*, \tag{3}$$

where indices "$j+$" and "$j-$" identify the jth pair of two opposite parallel boundary surfaces of a RUC.

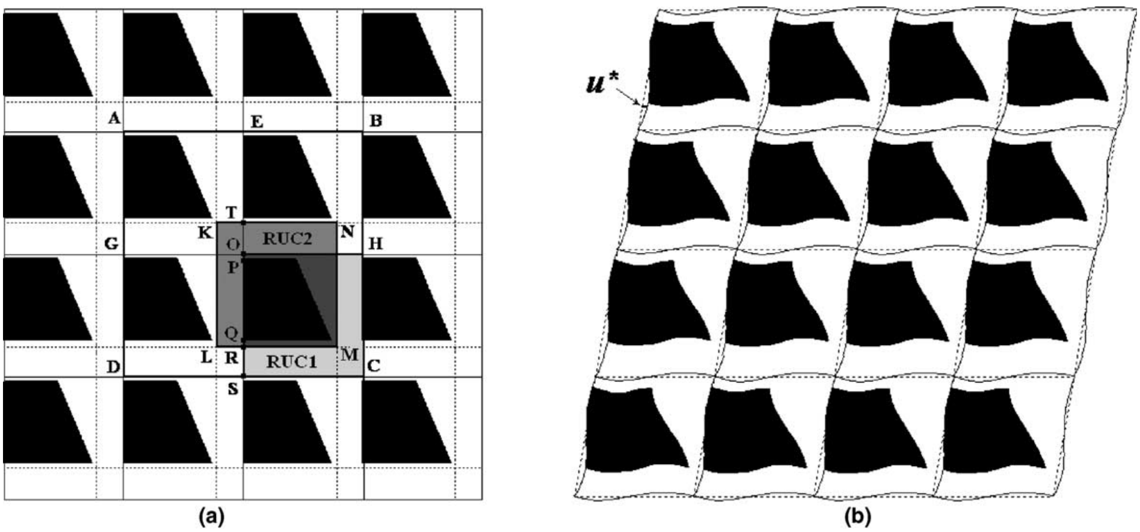

Fig. 1. A 2-D periodic structure without orthogonal axis of symmetry: (a) undeformed shape with two alternative choices of RUCs; (b) deformed shape under certain global load.

Note that $u_i^*(x_1, x_2, x_3)$ is the same at the two parallel boundaries (periodicity), therefore, the difference between the above two equations is

$$u_i^{j+} - u_i^{j-} = \bar{\varepsilon}_{ik}(x_k^{j+} - x_k^{j-}) = \bar{\varepsilon}_{ik}\Delta x_k^j. \quad (4)$$

Since Δ_k^j are constants for each pair of the parallel boundary surfaces, with specified $\bar{\varepsilon}_{ik}$, the right side becomes constants and such equations can be easily be applied in the finite element analysis as nodal displacement constraint equations (see Section 4 for more details about how to apply these constraint equations). Eq. (4) is a special type of displacement boundary conditions. Instead of giving known values of boundary displacements, it specifies the displacement-differences between two opposite boundaries. Obviously, the application of it will guarantee the continuity of displacement field. However, in general, such displacement-difference boundary conditions, Eq. (4), may not be complete or may not guarantee the traction continuity conditions. The traction continuity conditions can be written as

$$\sigma_n^{j+} - \sigma_n^{j-} = 0, \quad \tau_{nt}^{j+} - \tau_{nt}^{j-} = 0, \quad (5a, b)$$

where σ_n and τ_{nt} are normal and shear stresses at the corresponding parallel boundary surfaces, respectively. For general periodic boundary value problems the Eqs. (4) and (5) constitute a complete set of boundary conditions.

In the following, however, we will prove that if RUC is analyzed by using a *displacement-based finite element method*, the application of only Eq. (4) can guarantee the uniqueness of the solution and thus Eq. (5a,b) are automatically satisfied. In other word, the latter boundary conditions are not necessary to be applied in the analysis.

3. Uniqueness of solution in a displacement-based FEM when Eq. (4) is used

We first prove the uniqueness of the solution if only the displacement-difference boundary conditions, Eq. (4), are applied in a displacement-based FEM.

Assume that there exist two different solutions, u_i^{I} and u_i^{II}, both of them satisfying Eq. (4) at the corresponding pairs of parallel boundary surfaces of a RUC, i.e.,

$$(u_i^{\mathrm{I}})^{j+} - (u_i^{\mathrm{I}})^{j-} = \bar{\varepsilon}_{ik}\Delta x_k^j, \quad (6)$$

$$(u_i^{\mathrm{II}})^{j+} - (u_i^{\mathrm{II}})^{j-} = \bar{\varepsilon}_{ik}\Delta x_k^j. \quad (7)$$

Let

$$\bar{u}_i = u_i^{\mathrm{I}} - u_i^{\mathrm{II}}, \quad (8)$$

by subtracting Eq. (7) from Eq. (6), the displacement function $\bar{u}_i$ will meet the following trivial (zero) boundary conditions:

$$\bar{u}_i^{j+} - \bar{u}_i^{j-} = 0. \quad (9)$$

If one can show that $\bar{u}_i \equiv 0$, then the uniqueness of the solution will be proved.

It is not difficult to prove that in a displacement-based FEM analysis, only the trivial solution, $\bar{u}_i \equiv 0$, *will be obtained if the trivial boundary condition*, *Eq.* (9), *is applied to a RUC*:

In a displacement-based FEM the assembled global stiffness equation is expressed as

$$[K]\{U\} = \{F\}, \quad (10)$$

where $[K]$ is the system global stiffness matrix, $\{U\}$ and $\{F\}$ are the global nodal displacement and global nodal force vectors, respectively. A penalty approach (Chandrupatla and Belegundu, 2002) is used to impose the nodal displacement constraint equations. However, application of the constraint Eq. (9) in the FEM will only change certain elements in the stiffness matrix $[K]$ by adding large numbers to these elements, the original trivial force vector $\{F\} = \{0\}$ will not be changed since the right side of Eq. (9) is trivial (zero). Therefore, only the solution, $\{U\} \equiv \{0\}$, will be obtained as long as the stiffness matrix $[K]$ is not a singular one.

This can also be explained from the principle of minimum strain energy which is the basis of the displacement-based FEM: Since any deformed body has positive strain energy, among all possible solutions of Eq. (9), the trivial solution has the minimum strain energy (zero).

Thus we can conclude that

Theorem. *In a displacement-based FEM analysis, applying the unified displacement-difference periodic boundary conditions,* $u_i^{j+} - u_i^{j-} = \bar{\varepsilon}_{ik}\Delta x_k^j$, *on a RUC, a unique solution is obtained.*

Based on the uniqueness of the solution by applying the periodic boundary conditions, Eq. (4), in the displacement-based FEM analysis, the following two lemmas are not difficult to be obtained:

Lemma 1. *For a fixed periodic structure, different RUC's may be defined, however, by applying the unified displacement-difference periodic boundary conditions, Eq. (4) in the displacement-based FEM analysis, the solution will be independent of the choice of the RUCs.*

Note that in the above proof of the uniqueness of the solution, we have not specified the choice of any particular RUC for the periodic structure. The obtained unique displacement solution, Eq. (1), or the unique solution of $u_i^*(x_1, x_2, x_3)$ in Eq. (1) is for the entire periodic structure and therefore it is independent of the choice of the RUCs.

Lemma 2. *The solution obtained by applying the unified displacement-difference periodic boundary conditions, Eq. (4) in the displacement-based FEM analysis, will also meet the traction continuity conditions, Eq. (5a,b).*

Now consider a larger RUC (ADCB) consisting of four smaller RUCs shown in Fig. 1a. Application of Eq. (4) to the larger RUC will produce a unique and the same $u_i^*(x_1, x_2, x_3)$ as that for each smaller RUCs (quarter of the larger one). In the larger RUC (ADCB), the tractions are continuous at the line EO, which is the right boundary of the smaller RUC (AGOE) and the left boundary of smaller RUC (EOHB). Thus the tractions at the opposite boundaries, AG and EO for the RUC (AGOE) or at the opposite boundaries, EO and BH for the RUC (EOHB) are the same, i.e., Eq. (5a,b), should be satisfied.

It is to be emphasized that the above theorem and lammas all contain the prerequisite *in the displacement-based FEM analysis*. For general periodic boundary value problems the Eqs. (4) and (5) constitute a complete set of boundary conditions. If other solution methods (analytical or numerical) are used, only application of the Eq. (4) may not guarantee the uniqueness of the solution. The following counter-example shows that, in general, non-trivial solutions ($\bar{u}_i \neq 0$) could exist only if the Eq. (9) needs to be satisfied.

Counter-example. A 2-D uniform square is subjected to the following displacement boundary conditions (Fig. 2):

$$\bar{u} = \bar{v} = 0 \quad \text{at AB and DC},$$

$$\bar{v} = 0 \quad \text{at AD and BC},$$

$\bar{u}$ is given at AD and BC as shown in the Fig. 2a.

It is obvious that the above given displacement boundary conditions do satisfy the trivial Eq. (9). Fig. 2a together with Fig. 2b and c shows the non-trivial solution obtained by using the FEM code NASTRAN with the specified non-trivial displacement component $\bar{u}_x$, and the corresponding stress components σ_x

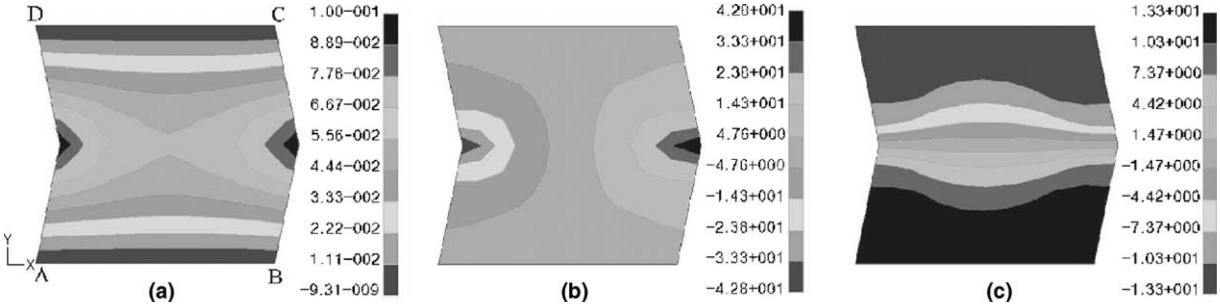

Fig. 2. A counter-example: a non-trivial solution which satisfies the trivial Eq. (9). (a) displacement $\bar{u}_x$; (b) stress σ_x; (c) stress τ_{xy}.

and τ_{xy}, respectively. (All FEM results in this paper are obtained from the NASTRAN and PATRAN code.) Obviously, this is a non-trivial solution for Eq. (9). In fact, there exists infinite number of such non-trivial solutions, which would satisfy the trivial Eq. (9). Therefore, adding any non-trivial solution of Eq. (9) to a solution of Eq. (4) will still be a solution of Eq. (4). This, in turn, confirms that for a general periodic boundary value problem, applying boundary conditions, Eq. (4), may not guarantee the uniqueness of the solution if the problem is to be solved by using methods other than the displacement-based FEM.

Now using the same FEM code, we directly apply the trivial boundary conditions, Eq. (9) to the above square, i.e.,

$$\bar{u}_{\mathrm{AB}} - \bar{u}_{\mathrm{DC}} = 0, \quad \bar{v}_{\mathrm{AB}} - \bar{v}_{\mathrm{DC}} = 0,$$

$$\bar{u}_{\mathrm{AD}} - \bar{u}_{\mathrm{BC}} = 0, \quad \bar{v}_{\mathrm{AD}} - \bar{v}_{\mathrm{BC}} = 0$$

and $\bar{u}_{\mathrm{A}} = \bar{v}_{\mathrm{A}} = 0$ (to eliminate the rigid body displacements).

Fig. 3 does show that in this case a trivial solution is obtained.

One advantage of the micro-mechanical analysis with the application of the above-mentioned unified periodic boundary conditions is that the entire stiffness matrix (or the flexibility matrix) can easily be obtained as soon as enough sets of the global strains are applied. Assuming a set of global strains, $\bar{\varepsilon}_{ij}$, and applying the periodic boundary conditions, Eq. (4), in the FEM analysis, one can obtain a unique solution including stress distributions. Then the global stresses, $\bar{\sigma}_{ij}$, corresponding to this set of global strains can be obtained through

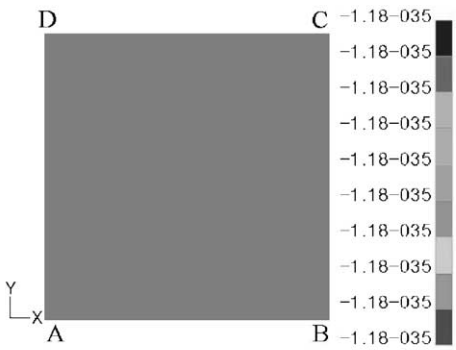

Fig. 3. A trivial solution, $\bar{u}_i \equiv 0$ is obtained by applying the trivial boundary conditions, Eq. (9), $\bar{u}_i^{j+} - \bar{u}_i^{j-} = 0$ in the displacement-based FEM analysis.

$$\bar{\sigma}_{ij} = \frac{1}{V}\int_V \sigma_{ij}\,\mathrm{d}V. \tag{11}$$

By using Gauss's theorem this volume integration can be trans formed to the surface integration over the entire boundary surfaces and thus the global stresses can be related to the ratios of resultant traction forces on the boundary surfaces to the corresponding areas of the boundary surfaces. For example, for a rectangular RUC, the following results can be obtained (see Xia et al., 2003):

$$\bar{\sigma}_{ij} = \frac{(P_i)_j}{S_j} \text{ (no summation over } j), \tag{12}$$

where S_j is the area of the jth boundary surface and $(P_i)_j$ is the ith resultant traction forces on the jth boundary surface.

The global stress-global strain relation of the periodic structure can be written as

$$\{\bar{\sigma}\} = [k]\{\bar{\varepsilon}\}. \tag{13}$$

For a 3-D case applying one set of $\bar{\varepsilon}_{ij}$ (six components), we thus obtain six equations. If the material is an orthotropic one, there are nine independent material constants in the stiffness matrix, $[k]$. One can apply another set of the $\bar{\varepsilon}_{ij}$ (the two strain vectors should be linearly independent, i.e., the ratios between the corresponding strain components of the two vectors are not the same). Then there will be enough equations to determine the nine material constants. In a more general case where there is no orthotropic axis of symmetry of the material, the application of four linearly independent sets of the $\bar{\varepsilon}_{ij}$ will be sufficient to determine 21 independent material constants in the $[k]$.

It is also noted that the proposed unified periodic boundary conditions are in the form of global strains. In the case of given global stresses, or a combination of the global stresses and strains, a proper proportion between the global strains must be applied. The required proportion can be determined without any difficulty through an iterative procedure, see Zhang et al. (2005).

One can also see that the derivation and proof procedures for the proposed unified periodic boundary conditions are not dependent on the properties of the constituent materials of the composites. Therefore, they can be applied to non-linear micro-mechanical analyses of the composites under any combination of multiaxial loads. An application example for non-linear viscoelastic analysis with given global multiaxial stress state can be found in Zhang et al. (2005).

4. Illustrative examples

For simplicity, all the following illustrative examples are 2–D plan stress state problems.

To apply the constraint equations (4) in FEM, it is better to produce the same meshing at each two paired boundary surfaces. Then each constraint equation in (4) contains only two displacement components of the paired nodes. The number of the constraint equations is usually quite large, certain preprocessing program can be used to produce the data depending on the individual FEM code used. It is noted that the proposed unified displacement-difference periodic boundary conditions are applied to the whole RUC. If certain symmetry exists in the RUC, a part of the RUC (a sub-unit of the RUC) can be used to reduce the size of the problem. It then requires solving the problem separately for the individual global strain/stress component and deriving corresponding boundary and symmetry conditions, for example, see Tang and Whitcomb (2003). However, with the current capacity of computer technology, it might not pose an obstacle to use a larger full RUC model but to apply the simpler unified constraint equations to solve the periodic boundary problems. In all following FEM analyses four node plane stress elements are used with small deformation assumption. The convergence of the solutions has been verified by comparing the results with different meshing sizes. It is found that relatively fine meshing size is required in order to obtain more accurate stress distribution, especially near the boundaries of the RUC. For example, the solutions presented in Section 4.1 are obtained by using 11,308 nodes and 11,508 elements. However, if only the global stiffness is concerned, relative coarse meshing size can still provide satisfactory results. For the same problem in Section 4.1, the error in predicted stiffness by using a meshing size of 356 nodes and 420 elements is within 3%, comparing to that by using the much finer meshing size.

4.1. Illustrative example I—comparison of proposed unified displacement-difference periodic boundary conditions and the homogeneous displacement boundary conditions

Consider the periodic structure as shown in Fig. 1. The volume fraction of the reinforcing phase is 45.5%. Assume both reinforcing and matrix phases are elastic and their material constants are

$$E_f = 200\ \text{GPa},\quad \nu_f = 0.2 \quad \text{and} \quad E_m = 2\ \text{GPa},\ \nu_m = 0.4.$$

Two smallest periodic RUCs are portrayed in Fig. 1a: RUC1 is identified by OSCH and RUC2 by KLMN.

Let us first choose the RUC1 (Fig. 1a) to carry out the FEM analysis and apply the constraint Eq. (4) with

$$\bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = 0.01 \text{ and } \bar{\varepsilon}_{12} = 0.$$

The solution is shown in Fig. 4. Note that for this example the assumption of "plane-remains-plane" at the boundaries is not even true for the global normal strains mode. From the stress results, one can also confirm that at the opposite parallel boundaries the normal and shear stresses are the same. Although only global normal strains are applied, the global shear stress is not zero because there is no orthogonal axis of symmetry for this particular periodic structure, or there exists normal-shear stress coupling. By using Eq. (12), the global stresses corresponding to the above global strain values are: $\bar{\sigma}_{11} = 72.59$ MPa, $\bar{\sigma}_{22} = 75.14$ MPa and $\bar{\tau}_{12} = 0.38$ MPa.

In contrast, Fig. 5 shows the results by applying the "plain-remains-plane" boundary conditions to this problem with the same values of the global normal strains. One can see that the normal stresses at the corresponding parallel boundaries are not the same, i.e., the traction continuity conditions, Eq. (5a,b), are violated and therefore this distribution of stresses cannot represent the real one of a physically continued periodic structure.

The matrix of stiffness, $[k]$, for the periodic structure with no orthogonal axis of symmetry is a full matrix, i.e., six independent elastic material constants are contained in the $[k]$ for the 2-D periodic structure. The $[k]$ can be determined by applying, for example, the following two sets of $\bar{\varepsilon}_{ij}$:

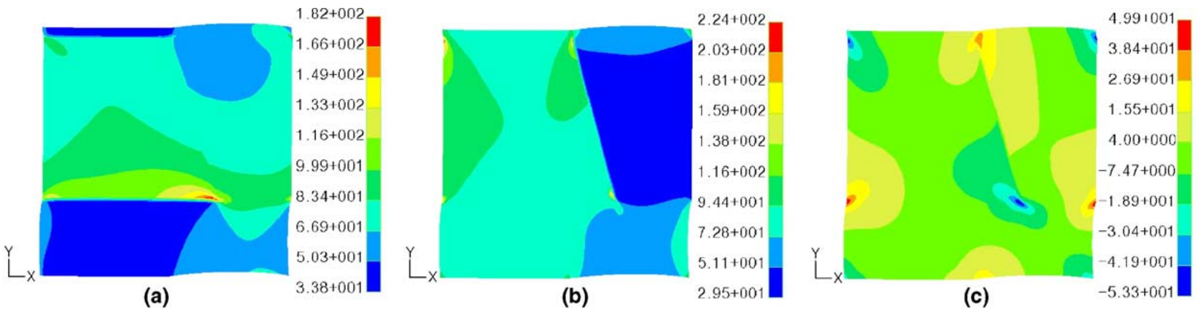

Fig. 4. FEM solution of RUC1 by applying the periodic boundary conditions with global strains of $\bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = 0.01$ and $\bar{\varepsilon}_{12} = 0$: (a) stress σ_x (MPa); (b) stress σ_y (MPa); (c) stress τ_{xy} (MPa).

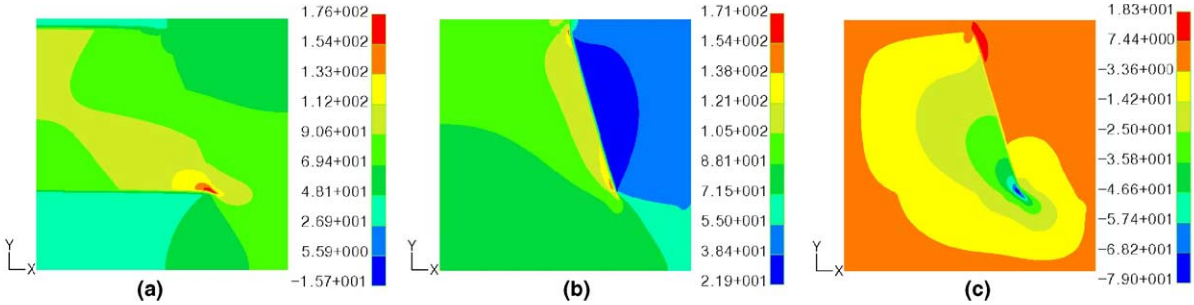

Fig. 5. FEM solution of RUC1 by applying the "plain-remains-plane" boundary conditions with global strains of $\bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = 0.01$ and $\bar{\varepsilon}_{12} = 0$: (a) stress σ_x (MPa); (b) stress σ_y (MPa); (c) stress τ_{xy} (MPa).

$$\bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = 0 \quad \text{and} \quad \bar{\varepsilon}_{12} = 0.01,$$

the corresponding $\bar{\sigma}_{ij}$ are obtained follows:

$$\bar{\sigma}_{11} = 0.18 \text{ MPa}, \quad \bar{\sigma}_{22} = 0.20 \text{ MPa} \quad \text{and} \quad \bar{\tau}_{12} = 14.55 \text{ MPa}.$$

Similarly

$$\bar{\varepsilon}_{22} = 0.01 \quad \text{and} \quad \bar{\varepsilon}_{11} = \bar{\varepsilon}_{12} = 0,$$

the corresponding $\bar{\sigma}_{ij}$ are obtained as

$$\bar{\sigma}_{11} = 13.40 \text{ MPa}, \quad \bar{\sigma}_{22} = 61.72 \text{ MPa} \quad \text{and} \quad \bar{\tau}_{12} = 0.20 \text{ MPa}.$$

From these results the stiffness and flexibility matrices for the periodic structure are determined as:

$$[k] = \begin{bmatrix} 5.916 & 1.343 & 0.018 \\ 1.343 & 6.171 & 0.020 \\ 0.018 & 0.020 & 1.455 \end{bmatrix} \text{GPa} \quad [f] = [k]^{-1} = \begin{bmatrix} 0.178 & -0.039 & -0.002 \\ -0.039 & 0.170 & -0.002 \\ -0.002 & -0.002 & 0.687 \end{bmatrix} \text{GPa}^{-1}.$$

One can also compare the stiffness matrix obtained based on the "plane-remains-plane" boundary conditions, which is the following:

$$[k] = \begin{bmatrix} 6.191 & 1.202 & 0.028 \\ 1.202 & 6.453 & 0.717 \\ 0.028 & 0.717 & 2.102 \end{bmatrix} \text{GPa}.$$

Obviously, a stiffer prediction is obtained and the error is even bigger for the shear modulus (44% higher).

It is seen from the above results that it is an easy and routine procedure to obtain the global stress/strain relations for the periodic structure through application of the suggested unified periodic boundary conditions to the RUC. For a non-linear micro-mechanical analysis, the instantaneous stiffness or flexibility matrix of the system can be determined in a similar way.

In the following we further show the independence of the solution on the choice of different RUCs for the same periodic structure by applying the proposed unified periodic boundary conditions.

4.2. Illustrative example II

For the same periodic structure indicated in Fig. 1, an alternative RUC is the RUC2 as shown in Fig. 1a. We repeat the FEM analysis for the RUC2 with the same applied global strains as that of the RUC1

$$\bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = 0.01 \quad \text{and} \quad \bar{\varepsilon}_{12} = 0.$$

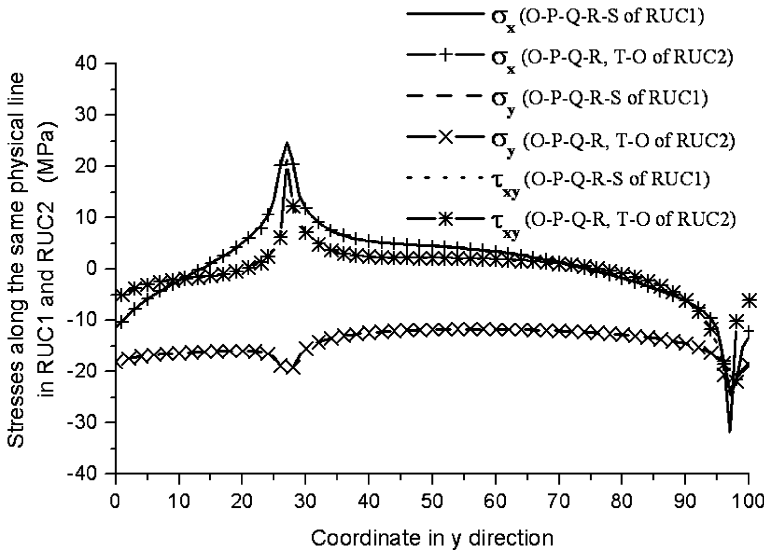

Fig. 6. The stress components along a same physical line OPQRST in the same periodic structure (Fig. 1a), from FEM analyses on different models, RUC1 and RUC2, respectively.

Fig. 6 shows the variations of the stress components along the same physical line in the structure from the two FEM analyses. For the RUC1 the line is indicated by letter array O–P–Q–R–S and for the RUC2 is identified by O–P–Q–R and T–O. It can be seen that exactly the same stress distributions are obtained for the two different RUCs.

4.3. Illustrative example III

The objective of this example is to further show that the solution is independent of the choice of RUCs even with different shapes. Consider the periodic structure as shown in Fig. 7. The volume fraction of the reinforcing phase is 64% and the material constants are taken to be the same as in the previous example. As shown in the Fig. 7, either the RUCs with the rhombohedral or the hexagonal shapes can be defined for this periodic structure. Fig. 8 shows the FEM results indicate exactly the same stress distribution along the same physical line AB of the two RUCs with different shapes.

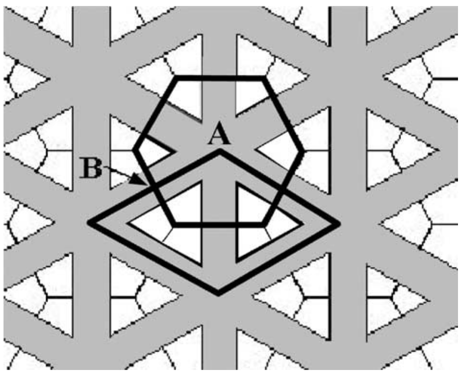

Fig. 7. A 2-D quasi-isotropic periodic structure and choice of two RUCs with different shapes.

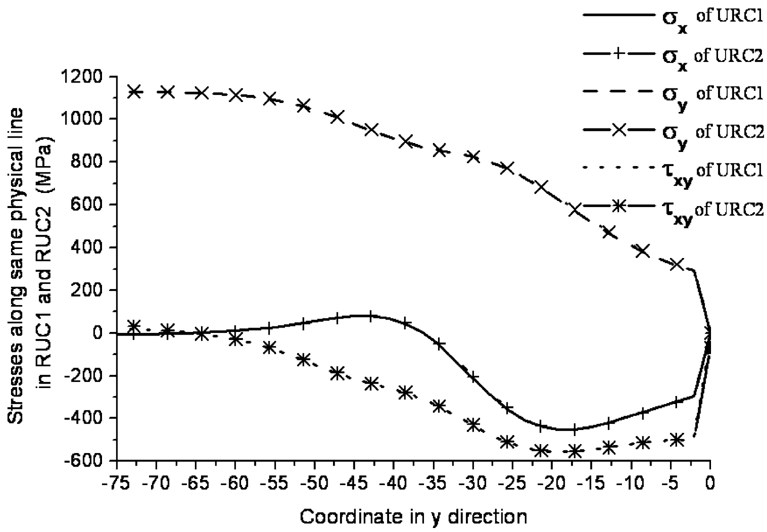

Fig. 8. The stress components along a same physical line AB in the same periodic structure (Fig. 7), from FEM analyses on two RUCs with different shapes.

Since this periodic structure has three axes of symmetry in the plane it must have a quasi-isotropic global mechanical property in this plane. Through applying a set of global strains, e.g., $\bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = \bar{\varepsilon}_{12} = 0.01$, the elastic modulus, Poisson's ratio and the shear modulus can be obtained as

$$E = 46.09\ \text{GPa}, \quad v = 0.302 \quad \text{and} \quad G = 17.70\ \text{GPa}.$$

One can see that the shear modulus calculated is equal to $G = E/2/(1 + v)$, which confirms the quasi-isotropic feature of the periodic structure.

In the above two examples since the stress results are the same from the different RUCs by applying the same global strains, the global stresses and therefore the stiffness matrices calculated will be the same.

5. Conclusions

An explicit unified form of displacement-difference periodic boundary conditions for repeated unit cell (RUC) model is presented. This type of boundary conditions can be easily applied in FEM analysis as a set of constraint equations of nodal displacements of corresponding nodes on the opposite parallel boundary surfaces of the RUC model.

The solution is unique by applying the proposed unified periodic boundary conditions on the RUC in a displacement-based FEM analysis.

For a fixed periodic composite structure the choice of RUC is not unique but identical solution can be obtained by applying the proposed unified periodic boundary conditions to any correctly defined RUC.

The application of the unified periodic boundary conditions can guarantee the displacement continuity and the traction continuity at the boundaries of the RUC model and as such it is the solution for a real periodic composite structure.

By applying enough sets of the global strains in the unified periodic boundary conditions, entire stiffness or flexibility matrix for a periodic composite structure can be predicted.

The proposed unified periodic boundary conditions can also be applied to non-linear micro-mechanical analysis of the composites under any combination of multiaxial loads.

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